ISRAEL JOURNAL OF MATHEMATICS. Vol 30, No 3, 1978

AN AXIOM SYSTEM FOR THE WEAK MONADIC SECOND ORDER THEORY OF TWO SUCCESSORS[†]

BY
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ABSTRACT

A complete axiom system for the weak monadic second order theory of two successor functions, W2S, is presented. The axiom system consists, roughly, of the generalized Peano axioms and of an inductive definition of the finite sets. For the proof, methods of J. R. Buchi and J. Doner are used to obtain a new decision procedure for W2S, whose proofs are easily formalized. Different finiteness axioms are discussed.

0. Introduction

Let W2S be the weak monadic second order theory of two successor functions, i.e. the theory of the full binary tree which allows quantification over both elements and finite subsets of the tree. Doner [4], and independently though somewhat later Thatcher–Wright [9], have shown that W2S is decidable. W2S is thus trivially axiomatizable by its true sentences. It is not just for aesthetical reasons, however, that this paper presents a "neat" axiom system for W2S. When working with monadic second order theories one actually works with fragments of set theory. Thus there is no absolute frame of monadic second order logic, and it is questionable whether there is such a frame even for only the decidable monadic second order theories. Therefore when proving the decidability of a monadic second order theory, one should specify what part of set theory one needs for the proof. (Compare Buchi–Siefkes [3].) The situation is less uncertain in the case of a weak monadic second order theory. Still there are

[†] This paper was written while the author was visiting at Purdue University, and appeared first as Report CSD TR-56, Purdue University, 1971.

Received February 20, 1977

different definitions of infinity, some of which are equivalent only by the axiom of choice. So one should find out which definition(s) one is using.

In case of a decidable theory, to reveal the very content of the theory one has to formalize the decision procedure, and under the way to collect all principles one needs for deciding. The result for W2S is a complete axiom system which consists of three parts: (i) Axioms for the elementary logic of this language. (ii) Axioms for the two successor functions, which are generalizations of the Peano axioms for one successor. (iii) Axioms characterizing the finite sets as an inductive structure generated from the empty set by the operation of adjoining an element. In other words, one gets exactly the finite sets by repeatedly adding single elements, starting from the empty set. This definition, however, is inductive and not explicit, since "repeatedly" means "finitely often".

As usual with monadic second order theories, Doner's decison procedure for W2S involves finite automata — tree automata in this case. It is his discovery that one has to have tree automata working backwards, from the branches to the root, in order to make deterministic automata useful. The same concept appears independently in Thatcher–Wright [9] and in Magidor–Moran [5]. The latter paper, however, is the only place where the problem is discussed explicitly: Magidor–Moran define "climbing" and "sinking" tree automata, and give an example of a finite (in fact, two-element) set of trees which is not definable by a climbing deterministic automaton.

The automata notions are easily formalized for our completeness proof. Doner's main proof tool, however, tree induction, cannot be expressed in the language of W2S; so we cannot formalize his proof. Instead of tree induction we use two other principles, induction for frontiered trees (see definition in section 3) and bar induction, the latter well known in intuitionistic mathematics. All these principles base on the fact that one can define the frontiered trees inductively: one gets exactly the frontiered trees by starting with the trivial root tree and adding repeatedly pairs $\{x0, x1\}$ of successors. Using these ideas we give a new decision procedure for W2S, which uses Doner's backward automata, but resembles in structure exactly the decision procedures for the weak and the strong monadic second order theory of one successor of Buchi [1] and [2], as presented in [8]. Thus in W2S we have a new case of the connection between monadic second order theories and finite automata, a connection which has proven to be stimulating in both directions.

I express my thanks to J. R. Buchi who greatly influenced this work and its presentation; some of his ideas can be found in this introduction. I also wish to thank J. Doner for many helpful discussions on the subject. I thank G. H. Müller

who originated and stimulated my interest in monadic second order theories. And I thank M. O. Rabin who introduced me into the problems of the binary tree.

1. The binary tree and the system W2S

Let T_2 be the set of all finite sequences of 0's and 1's. T_2 can best be pictured as the full binary tree, where the root represents the empty sequence e, and each element x has two successors, $x0$ and $x1$ (Fig. 1).

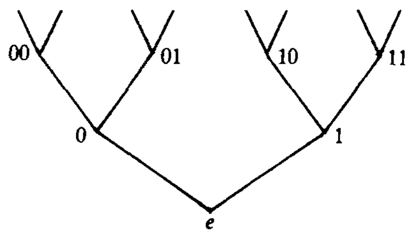

Fig. 1

The language to describe T_2 consists of individual variables $t, u, \cdots, z$, set variables $U, V, \cdots, Z$, the equality sign $=$, two unary function symbols s_0 and s_1, and an individual constant e. Prime formulae are of the form Xz, $x = y$, and $X = Y$. Arbitrary formulae are built up from prime formulae using sentential connectives and quantifiers for both types of variables. The interpretation of the formulae is suggested by the notation: individual variables range over the elements of T_2, set variables range over finite subsets of T_2, so Xz means "z is an element of X", the constant e denotes the root of the tree, and the two function symbols are used for the two successor functions. We stipulate that a sentence is *true* iff it is true in T_2 under this interpretation, and call the resulting system (= interpreted theory) W2S, *W*eak monadic second order theory of *2* *S*uccessors. This name is adapted from Rabin's S2S for the corresponding strong monadic second order theory of [6]. For the rest of the paper "set" will normally mean "finite subset of T_2".

In formulae of W2S we will use the following notation: We will write $x0$ and $x1$ instead of $s_0(x)$ and $s_1(x)$ respectively, especially 0 and 1 for $s_0(e)$ and $s_1(e)$. Also we will often write $z \in X$ instead of Xz, and we will use freely the usual set-theoretical notation, e.g. $z \notin X \cup \{y\}$ stands for $\neg[Xz \vee z = y]$. Similarly, sets and functions of elements or subsets of T_2 will be defined by comprehension; it should be kept in mind that thus defined terms are used only as abbreviations for expressions of the formal language. We will use Greek capital letters to denote formulae of W2S. The symbol $\equiv$ will denote literal equality of formulae in defining abbreviations.

2. The axiom system

The purpose of this paper is to show that the following three sets of axioms together constitute a complete axiom system (for derivability) for W2S, i.e. exactly the true sentences of W2S are derivable from this axiom system.

Part A

An arbitrary axiom system for the elementary logic of W2S, regarded for the moment as a two-sorted elementary theory. Here we add further two equality axioms:

(LEIBNIZ EQUALITY) $(\forall Z)[Zx \rightarrow Zy] \rightarrow x = y$,

(EXTENSIONALITY) $(\forall z)[Xz \leftrightarrow Yz] \rightarrow X = Y$.

Part B

Generalized Peano axioms for the two successor functions:

(OE1) $x0 \neq e$
(OE2) $x1 \neq e$
(the root has no predecessors),

(SE1) $x0 = y0 \rightarrow x = y$
(SE2) $x1 = y1 \rightarrow x = y$
(SE3) $x0 \neq y1$
(branches do not merge),

(IE) $\Phi(e) \wedge (\forall z)[\Phi(z) \rightarrow \Phi(z0) \wedge \Phi(z1)] \rightarrow (\forall z)\Phi(z)$

(induction schema for elements of the tree).

Part C

Axioms for finite subsets of the tree:

(OS) $(\exists X)X = \varnothing$ (existence of the empty set),

(SS) $(\forall X)(\forall y)(\exists Z)Z = X \cup \{y\}$

(the union of a set with a singleton is a set),

(IS) $\Phi(\varnothing) \wedge (\forall Z)(\forall x)[\Phi(Z) \rightarrow \Phi(Z \cup \{x\})] \rightarrow (\forall Z)\Phi(Z)$

(induction schema for subsets of the tree).

Remarks. The axiom system of part A consists of axioms for a two-sorted first order predicate calculus restricted to our language. An example of such an axiom system may be found on p. 4/5 of the author's [8], if one (i) changes the substitution rule (SP) for predicate variables into a rule for changing free set variables, and (ii) adds the usual equality axioms which make = a congruence relation. (SP) must be deleted, since it is equivalent to the full comprehension principle, and thus is false in W2S. We will instead derive the comprehension principle restricted to finite sets. There seems to be no finiteness definition in W2S which would allow us to replace part C by this restricted comprehension principle and the axiom that all sets are finite. (See the discussion following Proposition 3.6.)

If we had not included equality as a primitive notion, we could define it as usual:

$$x = y \leftrightarrow (\forall Z)[Zx \leftrightarrow Zy],$$

$$X = Y \leftrightarrow (\forall z)[Xz \leftrightarrow Yz].$$

These two equivalences are derivable from part A, and have to be used to get the formal counterparts for the axioms of part C.

Main Theorem. *Exactly the true sentences of W2S are derivable from the above axioms.*

It is easy to see that all the axioms are true in W2S. It remains to show that the axioms are complete, i.e. that all true sentences are derivable. To prove this we will (i) describe a decison procedure by which any sentence of W2S is transformed into an equivalent truth value, and (ii) show at the same time that the equivalences in the single steps of the procedure are derivable. The axioms are thus complete in an effective sense: for any true sentence of W2S, we can find effectively a derivation.

3. Basic properties of the tree

We will try to get some insight into the basic structure of T_2, which will prove valuable later in the decision procedure. For the rest of the paper, in all lemmata, propositions and theorems, the reader should add the phrase "The following is derivable from the axioms". Normally, however, the proofs will be given in a half-formal way, only indicating how a derivation could be built up.

We start by deriving two versions of the comprehension principle for finite sets, stating that definable parts of sets are sets.

Proposition 3.1.

$(\mathrm{COMP}_{\mathrm{fin}})$ $\quad (\exists W)(\forall x)[\Phi(x) \rightarrow Wx] \rightarrow (\exists U)(\forall x)[Ux \leftrightarrow \Phi(x)],$

$(\mathrm{COMP}^*_{\mathrm{fin}})$ $\quad (\exists U)(\forall x)[Ux \leftrightarrow Wx \wedge \Phi(x)].$

Proof. It is easy to derive the equivalence of the two forms. We will derive $(\mathrm{COMP}^*_{\mathrm{fin}})$: Let

$$\psi(W) \underset{\mathrm{df}}{\equiv} (\exists U)(\forall x)[Ux \leftrightarrow Wx \wedge \Phi(x)].$$

We will show $(\forall W)\psi(W)$ with the help of the set induction schema (IS). $\psi(\emptyset)$ means actually

$$(\forall Y)[(\forall x)\neg Yx \rightarrow \psi(Y)].$$

Now by predicate logic we have

$$(\forall x)\neg Yx \wedge (\forall x)\neg Ux \rightarrow (\forall x)[Ux \leftrightarrow Yx \wedge \Phi(x)],$$

and thus

$$(\forall x)\neg Yx \wedge (\exists U)(\forall x)\neg Ux \rightarrow (\exists U)(\forall x)[Ux \leftrightarrow Yx \wedge \Phi(x)].$$

Axiom (OS) yields

$$(\forall x)\neg Yx \rightarrow \psi(Y), \qquad \text{i.e.} \quad \psi(\emptyset).$$

$\psi(W \cup \{z\})$ is correctly expressed as

$$(\forall Y)\{(\forall u)[Yu \leftrightarrow u = z \vee Wz] \rightarrow \psi(Y)\}.$$

Then $\psi(W) \rightarrow \psi(W \cup \{z\})$ is derived by similar steps, which yields $(\forall W)\psi(W)$ by (IS). Q.E.D.

It is most important for our purpose that the natural partial order of T_2 be definable in W2S.

DEFINITIONS.

1) $\mathrm{Trans}(U) \equiv_{\mathrm{df}} (\forall z)[Uz0 \vee Uz1 \rightarrow Uz]$; U is *transitive*, i.e. closed under predecessor.

2) $x \leqq y \equiv_{\mathrm{df}} (\forall U)[\mathrm{Trans}(U) \wedge Uy \rightarrow Ux]$.

3) $x < y \equiv_{\mathrm{df}} x \leqq y \wedge x \neq y$.

4) $x \sim y \equiv_{\mathrm{df}} x \leqq y \vee y \leqq x$, x and y are *comparable*.

PROPOSITION 3.2. *$\leqq$ and $<$ are the natural partial orderings of T_2 induced by the successor functions, i.e.*

(a) $x \leqq x$,

(b) $x \leqq y \wedge y \leqq z \rightarrow x \leqq z$,

(c) $\neg x < x$,

(d) $x < y \wedge y < z \rightarrow x < z$,

(e) $x < y \rightarrow y \not\leqq x$,

(f) $x \leqq y \wedge y \leqq x \rightarrow x = y$,

(g) $e \leqq x$,

(h) $x < x0 \wedge x < x1$,

(i) $x < y \rightarrow x0 \leqq y \vee x1 \leqq y$,

(j) $x0 \leqq y \wedge x1 \leqq z \rightarrow y \not\sim z$.

The proof consists of a long chain of lemmata, which we will not give here. The interested reader should note that (e) is best derived before (d) and (f). It should be remarked that Proposition 3.2 is a consequence of the axioms of parts A and B alone, together with ($\mathrm{COMP}_{\mathrm{fin}}$).

The following definitions concern subsets of T_2. The terminology is partially taken from Doner [4] and Rabin [6]. Since in W2S we deal only with finite sets, however, the definitions of "frontier" and "frontiered tree" are different from Rabin's definitions for S2S, although the notions are the same; the notion of "path" is weaker than in S2S. Doner, on the other hand, calls "frontier" what we call "border"; so not every frontier in Doner's sense is a frontier in our sense.

DEFINITIONS.

1) $T_x =_{\mathrm{df}} \{y; x \leqq y\}$: the *tree with root x*.

2) $P_x =_{\mathrm{df}} \{y; y \leqq x\}$: the *path up to x*.

3) $\mathrm{Cl}(U) =_{\mathrm{df}} \{x; (\exists y \geqq x) Uy\}$: the *transitive closure* of U.

4) $\mathrm{Br}(U) =_{\mathrm{df}} \{x \in U; x0 \notin \mathrm{Cl}(U) \vee x1 \notin \mathrm{Cl}(U)\}$: the *border* of U.

5) $\mathrm{Br}^+(U) =_{\mathrm{df}} \{x \notin \mathrm{Cl}(U); (\forall y < x) y \in \mathrm{Cl}(U)\}$: the *outer border* of U.

6) $U^+ =_{\mathrm{df}} \mathrm{Cl}(U) \cup \mathrm{Br}^+(U)$: the *outer closure* of U.

7) $U^- =_{\mathrm{df}} U - \mathrm{Br}(U)$: the *interior* of U.

8) $\mathrm{Fr}(U) \equiv_{\mathrm{df}} (\forall z)(\exists y \in U) y \sim z \wedge (\forall y, z \in U)[z \sim y \rightarrow z = y]$: U is a *frontier*.

9) $\mathrm{FrTr}(U) \equiv_{\mathrm{df}} \mathrm{Trans}(U) \wedge \mathrm{Fr}(\mathrm{Br}(U))$: U is a *frontiered tree*.

10) $\mathrm{Fin}(U) \equiv_{\mathrm{df}} (\exists W)[\mathrm{Fr}(W) \wedge U \subseteq \mathrm{Cl}(W)]$: U is *finite*.

11) $\mathrm{Path}(U) \equiv_{\mathrm{df}} U \neq \varnothing \wedge \mathrm{Trans}(U) \wedge (\forall z)[Uz0 \rightarrow \neg Uz1]$: U is a *path*.

It should be remarked once more that definitions 1–11 are mere abbreviations, e.g. $y \in T_x$ means formally just $x \leqq y$; but T_x is not a set of our model. We will use notation like $U \in \mathrm{Trans}$ instead of $\mathrm{Trans}(U)$.

An example might illustrate these definitions (Fig. 2).

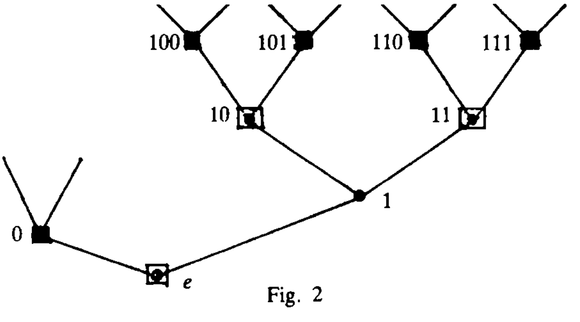

Fig. 2

Here the following sets occur:

● $U = \{10, 11\}$,
○,● $\mathrm{Cl}(U) = \{e, 1, 10, 11\}$,
□ $\mathrm{Br}(\mathrm{Cl}(U)) = \{e, 10, 11\}$,
■ $\mathrm{Br}^+(U) = \{0, 100, 101, 110, 111\}$.

The reader will more easily understand the proofs of this paper if he draws similar pictures.

Up to now we know only that the empty set exists; we will show now that there are a lot more sets.

PROPOSITION 3.3. *Existence of sets*:
(a) $(\forall x)(\exists U) \quad U = \{x\}$ (*singleton*),
(b) $(\forall x)(\exists U) \quad U = P_x$ (*path*),
(c) $(\forall W)(\exists U) \quad U = \mathrm{Br}(W)$ (*border*),
(d) $(\forall V, W)(\exists U) \quad U = V \cap W$ (*intersection*),
(e) $(\forall V, W)(\exists U) \quad U = V \cup W$ (*union*),
(f) $(\forall W)(\exists U) \quad U = \mathrm{Cl}(W)$ (*closure*),
(g) $(\forall W)(\exists U) \quad U = \mathrm{Br}^+(W)$ (*outer border*),
(h) $(\forall W)(\exists U) \quad U = W^+$ (*outer closure*),
(i) $(\forall W)(\exists U) \quad U = W^-$ (*interior*).

PROOF. We get (a) by combining (OS) and (SS). Using the formula

$$\Phi(x) \underset{\mathrm{df}}{\equiv} (\exists U) U = P_x,$$

we prove (b) by (IE), using (a) and (SS). (c), (d), and (i) are instances of ($\mathrm{COMP}_{\mathrm{fin}}$). Set induction (IS) on the formula

$$\Phi(W) \underset{\mathrm{df}}{\equiv} (\forall V)(\exists U) U = V \cup W$$

gives (e) with the help of (SS). (f) follows from (b) and (e) by (IS), if we note that

$$\mathrm{Cl}(U \cup \{x\}) = \mathrm{Cl}(U) \cup P_x.$$

We cannot yet prove (g); a proof follows from Proposition 3.4(a) together with (a), (e), and ($\mathrm{COMP}_{\mathrm{fin}}$). (h) finally is a direct consequence of (f), (g), and (e).

Q.E.D.

The next proposition provides two other induction schemata for sets:

PROPOSITION 3.4. (a) *Induction for transitive sets*:

$$\Phi(\varnothing) \wedge \Phi(\{e\}) \wedge (\forall U \in \mathrm{Trans})(\forall x \in U)[\Phi(U) \rightarrow$$
$$\rightarrow \Phi(U \cup \{x0\}) \wedge \Phi(U \cup \{x1\})] \rightarrow (\forall U \in \mathrm{Trans})\Phi(U).$$

(b) *Induction for frontiered trees*:

$$\Phi(\{e\}) \wedge (\forall U \in \mathrm{FrTr})(\forall x \in U)[\Phi(U) \rightarrow \Phi(U \cup \{x0, x1\})] \rightarrow (\forall U \in \mathrm{FrTr})\Phi(U).$$

PROOF. (a) Let the hypothesis be given, define

$$\psi(U) \underset{\mathrm{df}}{\equiv} \Phi(\mathrm{Cl}(U)).$$

We will show $(\forall U)\psi(U)$ by set induction (IS). Since for transitive U, $U = \mathrm{Cl}(U)$, we get the wanted conclusion

$$(\forall U \in \mathrm{Trans})\Phi(U).$$

$\mathrm{Cl}(\varnothing) = \varnothing$, therefore $\psi(\varnothing)$ holds. Let U be given such that $\psi(U)$. We will show $(\forall x)\psi(U \cup \{x\})$ by induction (IE). Start with $x = e$:

Case 1. $U = \varnothing$. Then $U \cup \{e\} = \{e\} = \mathrm{Cl}(\{e\})$, thus $\psi(U \cup \{e\})$ follows from $\Phi(\{e\})$.

Case 2. $U \neq \varnothing$. Then $e \in \mathrm{Cl}(U)$, therefore $\mathrm{Cl}(U \cup \{e\}) = \mathrm{Cl}(U)$, which together with $\psi(U)$ implies $\psi(U \cup \{e\})$.

Now let $\psi(U \cup \{x\})$ be proven; we have to show

$$\psi(U \cup \{x0\}) \wedge \psi(U \cup \{x1\}).$$

Since

$$\mathrm{Trans}(\mathrm{Cl}(U \cup \{x\})) \wedge x \in \mathrm{Cl}(U \cup \{x\}) \wedge \Phi(\mathrm{Cl}(U \cup \{x\})),$$

we get from the hypothesis of the proposition

$$\Phi(\mathrm{Cl}(U \cup \{x\}) \cup \{x0\}).$$

But

$$\mathrm{Cl}(U \cup \{x\}) \cup \{x0\} = \mathrm{Cl}(U \cup \{x0\}),$$

which implies $\psi(U \cup \{x0\})$. Analogously we get $\psi(U \cup \{x1\})$.

(b) Use $\psi(U) \equiv_{\mathrm{df}} \Phi(U^+)$ to prove $(\forall U)\Phi(U^+)$ by (IS) similarly as in (a). Let W be a frontiered tree, and let $U = W^-$. By Proposition 3.5(o) below, $U^+ = W$, which proves $\Phi(W)$. Q.E.D.

Now we can get more insight into the notions defined above.

PROPOSITION 3.5.

(a) $\mathrm{Path}(P_x)$,

(b) $[y \in P_x \wedge y \neq x \rightarrow y0 \in P_x \vee y1 \in P_x]$,

(c) *Paths are well-ordered*,

(d) $\mathrm{Br}(U) = U \cap \mathrm{Br}(\mathrm{Cl}(U))$, $\mathrm{Br}^+(\mathrm{Cl}(U)) = \mathrm{Br}^+(U)$, $(\mathrm{Cl}(U))^+ = U^+$,

(e) $\mathrm{Trans}(U^+)$,

(f) $x \in U \rightarrow (\exists y \in \mathrm{Br}(U))\ x \leqq y$,

(g) $\mathrm{FrTr}(U) \rightarrow [Ux0 \leftrightarrow Ux1]$,

(h) $W \subseteq U \rightarrow \mathrm{Br}^+(W) \subseteq \mathrm{Br}^+(U) \cup \mathrm{Cl}(U)$,

(i) $\mathrm{Br}(U^+) = \mathrm{Br}^+(U)$,

(j) $\mathrm{Fr}(\mathrm{Br}^+(U))$,

(k) $\mathrm{FrTr}(U^+)$,

(l) $(U^+)^- = \mathrm{Cl}(U)$,

(m) $\mathrm{Trans}(U) \leftrightarrow U = (U^+)^-$,

(n) $\mathrm{FrTr}(U) \leftrightarrow \mathrm{Trans}(U^-) \wedge \mathrm{Br}(U) = \mathrm{Br}^+(U^-)$,

(o) $\mathrm{FrTr}(U) \leftrightarrow U = (U^-)^+$.

PROOF. (a), (b), and (c) follow easily from Proposition 3.2. (d) follows directly from the definitions. (e), (g), and (h) are easy. To prove (f) let $x \in U$ be given. Let $W =_{\mathrm{df}} U \cap (T_x - \{x\})$. Then $\mathrm{Br}(W) = \mathrm{Br}(U) \cap (T_x - \{x\})$.

Case 1. $W = \varnothing$. Then $x \in \mathrm{Br}(U)$.

Case 2. $W \neq \varnothing$. It is easy to prove by set induction that

$$V \neq \varnothing \rightarrow \mathrm{Br}(V) \neq \varnothing.$$

So let $y \in \mathrm{Br}(W)$. Then $y \in \mathrm{Br}(U)$, and $x < y$.

Let us prove (i). $x \in \mathrm{Cl}(U)$ implies that

$$[x0 \in \mathrm{Cl}(U) \vee x0 \in \mathrm{Br}^+(U)] \wedge [x1 \in \mathrm{Cl}(U) \vee x1 \in \mathrm{Br}^+(U)]$$

and thus

$$x0 \in U^+ \wedge x1 \in U^+.$$

On the other hand, $x \in \mathrm{Br}(U^+)$ implies by (e) and by definition

$$x0 \notin U^+ \vee x1 \notin U^+.$$

Therefore, $x \in \mathrm{Br}(U^+)$ implies $x \notin \mathrm{Cl}(U)$, and trivially, $x \in U^+$, thus $x \in \mathrm{Br}^+(U)$. Conversely, let $x \in \mathrm{Br}^+(U)$. Then $x \in U^+$. It remains to show

$$x0 \notin U^+ \vee x1 \notin U^+.$$

Assume $x0 \in U^+$:

Case 1. $x0 \in \mathrm{Cl}(U)$. Then $x \in \mathrm{Cl}(U)$, thus $x \notin \mathrm{Br}^+(U)$, contradiction.

Case 2. $x0 \in \mathrm{Br}^+(U)$. Then $x \in \mathrm{Cl}(U)$, again contradiction. Thus $x0 \notin U^+$.

To prove (j), use Proposition 3.4(a), induction for transitive sets: $\mathrm{Br}^+(\emptyset) = \{e\}$, $\mathrm{Br}^+(\{e\}) = \{0,1\}$; thus the induction beginning is easy. Now let U be transitive, let $x \in U$, suppose $\mathrm{Fr}(\mathrm{Br}^+(U))$ as induction hypothesis. We have to show

$$\mathrm{Fr}(\mathrm{Br}^+(U \cup \{x0\})) \wedge \mathrm{Fr}(\mathrm{Br}^+(U \cup \{x1\})).$$

Case 1. $x0 \in U$. Then $U \cup \{x0\} = U$.

Case 2. $x0 \notin U$. Then $x \in \mathrm{Br}(U) \wedge x0 \in \mathrm{Br}^+(U)$. Thus

$$\mathrm{Br}^+(U \cup \{x0\}) = (\mathrm{Br}^+(U) - \{x0\}) \cup \{x00, x01\}.$$

Let $y, z \in \mathrm{Br}^+(U \cup \{x0\})$, $y \neq z$. To show: $y \not\sim z$. If $y, z \in \mathrm{Br}^+(U)$, then $y \not\sim z$ by induction hypothesis. If $y = x00$, $z = x01$, or conversely, then $y \not\sim z$. If, say, $y \in \mathrm{Br}^+(U) - \{x0\}$, $z = x00$, then $y \not\sim x0$, thus $y \not\sim z$. This proves the second clause in the definition of a frontier. To get the first clause, let y be given, let z be such that $z \in \mathrm{Br}^+(U) \wedge y \sim z$ (by induction hypothesis). We have to show

$$(\exists z \in \mathrm{Br}^+(U \cup \{x0\})) \quad y \sim z.$$

If $z = x0$, then $x00 \sim y \vee x01 \sim y$. If $z \neq x0$, then $z \in \mathrm{Br}^+(U \cup \{x0\})$. Thus we have shown

$$\mathrm{Fr}(\mathrm{Br}^+(U \cup \{x0\})).$$

The proof for $x1$ is analogous. Thus we have proved (j) for transitive sets which by (d) is enough. (k) follows from (e), (i), and (j). To prove (l) note that

$$(U^+)^- = U^+ - \mathrm{Br}(U^+) = U^+ - \mathrm{Br}^+(U) = \mathrm{Cl}(U)$$

by (i) and definition. (m) is a direct consequence of (l), since

$$\mathrm{Trans}(U) \leftrightarrow U = \mathrm{Cl}(U).$$

For (n) let $\mathrm{FrTr}(U)$. The transitivity of U^- follows easily from (f). So let $x \in \mathrm{Br}^+(U^-)$, thus $x \notin U^- \wedge (\forall y < x)\ y \in U^-$. Since $\mathrm{Fr}(\mathrm{Br}(U))$, there is $z \in \mathrm{Br}(U)$ such that $x \sim z$. If $z < x$ were true, then $z \in U^-$, which contradicts $z \in \mathrm{Br}(U)$. Thus $x \leqq z$, and therefore by the transitivity of $U, x \in U$. Since $x \notin U^-$, we have $x \in \mathrm{Br}(U)$. Thus

$$\mathrm{Br}^+(U^-) \subseteq \mathrm{Br}(U).$$

The converse inclusion is proved similarly. Now let conversely

$\mathrm{Trans}(U^-) \wedge \mathrm{Br}(U) = \mathrm{Br}^+(U^-)$. This implies easily the transitivity of U, whereas $\mathrm{Fr}(\mathrm{Br}(U))$ follows directly from (j).

For (o) again let $\mathrm{FrTr}(U)$:

$$(U^-)^+ = \mathrm{Cl}(U^-) \cup \mathrm{Br}^+(U^-) = U^- \cup \mathrm{Br}(U) = U$$

by (n). The converse direction follows from (k). Q.E.D.

PROPOSITION 3.6. *Finiteness principles (two others are to be found in Proposition* 3.5(j) (j) *and* (k)).

(a) $(\forall U \in \mathrm{Path})(\exists x)\ U = Px$: *every path has a maximum.*

(b) $(\forall U)\mathrm{Fin}(U)$: *every set is finite.*

(c) $(\forall U \in \mathrm{Fr})[U = \{e\} \vee (\exists x)[Ux0 \wedge Ux1]]$: *every frontier has local maxima.*

PROOF. (a) and (b) are easily proven by Proposition 3.4(a), induction for transitive sets, on the formulae

$$\mathrm{Path}(U) \rightarrow (\exists x) \quad U = Px$$

and $\mathrm{Fin}(U)$, respectively.

(c) is trivial with help of induction for frontiered trees, Proposition 3.4(b). Q.E.D.

It is easy to see that all five finiteness principles of Proposition 3.6 would be false in the strong monadic second order theory of two successors, Rabin's S2S, i.e. they don't hold in the tree with arbitrary subsets. But some of them are true in intermediate structures which have some infinite sets. Indeed, call a path U *infinite* iff

$$(\forall x \in U)[Ux0 \vee Ux1],$$

call a set W *thin* iff

$$(\forall U)[\text{Infinite path } (U) \rightarrow (\exists y \in U) W \cap U \cap T_y = \varnothing].$$

All finite sets are thin, but so is e.g. the *comb*, i.e. the set $\{0^n1;\ n < \omega\}$.

Now, in the structure which admits all thin sets but no other ones, Proposition 3.5(j) and (k) and Proposition 3.6(a) and (b) are true, but Proposition 3.6(c) is false. This shows that the following self-suggesting axiom system is not complete: Axioms parts A and B, together with $(\mathrm{COMP}_{\mathrm{fin}})$ and the finiteness axiom Proposition 3.6(b). (In the case of one successor function, the analogous axiom system is complete, since there the finite sets are uniquely defined as the bounded sets. See [8], pp. 117 ff.) The thin sets are not nice anyway, since the closure of a thin set need not be thin, and thus Proposition 3.3(f) and (h) are false. But if we enlarge the structure under consideration by taking the closure of sets, Proposition 3.6(a) becomes false, too, whereas the three other principles remain true. It might be that the above axiom system becomes complete if we add (a) and (c) as axioms (or just (c)?). We did not, however, investigate this question.

The stronger theory S2S is more powerful here. In S2S, a path is what we called above an infinite path (see Rabin [6]). And a frontier is then defined as a set which meets every path in exactly one point. With this concept of a frontier, the definition of a finite set as a set bounded by a frontier works properly. But in W2S, where infinite paths are not available, "finite" does not mean "bounded", but "built up point by point".

As an application of the finiteness principles we prove now two lemmata which we will need later in the decision procedure:

LEMMA 3.7.

$$(\exists x)\Phi(x) \leftrightarrow (\exists W)[We \wedge (\forall x \in W)[\neg Wx0 \wedge \neg Wx1 \to \Phi(x)]].$$

PROOF. $\to$: Let $\Phi(x)$ be true. It is easy to see that $W = P_x$ satisfies the lemma.

$\leftarrow$: Let W have the stated properties. $e \in W$ implies that $e \notin \mathrm{Br}^+(W)$. Therefore, by Proposition 3.5(j) and Proposition 3.6(c), there is an x such that

$$x0 \in \mathrm{Br}^+(W) \wedge x1 \in \mathrm{Br}^+(W).$$

Thus $\Phi(x)$ holds. Q.E.D.

Different forms of the induction principle for elements are easy consequences of Proposition 3.2, e.g. the minimum principle, stating that every set has a minimal element (it can have more than one, of course), or the maximum principle for subsets of a path. The following induction principle is different, and will be useful for the handling of tree automata. It is a special form of bar induction, used in intuitionistic mathematics, and is a consequence of the finiteness principle of Proposition 3.6(c):

LEMMA 3.8. *Bar induction*:

$$\mathrm{Trans}(U) \wedge (\forall x \in \mathrm{Br}^+(U))\Phi(x) \wedge (\forall x \in U)[\Phi(x0) \wedge \Phi(x1) \to \Phi(x)]$$
$$\to (\forall x \in U^+)\Phi(x).$$

PROOF. Let U, Φ satisfy the hypothesis of the lemma. By ($\mathrm{COMP_{fin}}$), there is a set W such that

$$(\forall x)[Wx \leftrightarrow Ux \wedge \neg \Phi(x)].$$

Assume $W \neq \varnothing$. By Proposition 3.6(c) there exists x such that

$$x0 \in \mathrm{Br}^{+}(W) \wedge x1 \in \mathrm{Br}^{+}(W),$$

and thus

$$x \in W, \quad x0 \notin W, \quad x1 \notin W.$$

Since $W \subseteq U$, and U is transitive, by Proposition 3.5(h) we have

$$\mathrm{Br}^{+}(W) \subseteq \mathrm{Br}^{+}(U) \cup U,$$

therefore

$$x0, x1 \in \mathrm{Br}^{+}(U) \cup U.$$

If $x0 \in \mathrm{Br}^{+}(U)$, then by hypothesis $\Phi(x0)$ holds. If $x0 \notin \mathrm{Br}^{+}(U)$, then $x0 \in U$, and therefore again $\Phi(x0)$ holds (by definition of W, since $x0 \notin W$). Analogously one gets $\Phi(x1)$, and therefore by hypothesis $\Phi(x)$. But this implies that $x \notin W$, contradiction. Q.E.D.

4. Tree automata, recursion and normal forms

Let Σ_n be the set of all n-tuples of truth values T, F. Our definition of tree automata is about the same as Doner's [4], but we choose both, the set of states and the input alphabet, among the Σ_n's. Our terminology is partially that of Rabin [6].

DEFINITIONS.

1) A *Σ_n-tree* is a function from a finite transitive subset of T_2 into Σ_n.

2) A *deterministic tree automaton over the alphabet Σ_n is a quadruple* $\mathfrak{A} = \langle \Sigma_k, s_0, J, K \rangle$, where Σ_k is the *set of states*, $s_0 \in \Sigma_k$ is the *initial state*, $J: \Sigma_n \times \Sigma_k \times \Sigma_k \to \Sigma_k$ is the *transition function*, and $K \subseteq \Sigma_k$ is the *set of final states*. The *run* of $\mathfrak{A}$ over the Σ_n-tree $X: U \to \Sigma_n$ as *input* is the Σ_k-tree $Z: U^{+} \to \Sigma_k$ defined by

$$(\forall t \in \mathrm{Br}^{+}(U))Zt = s_0 \wedge \ (\forall t \in U)Zt = J(Xt, Zt0, Zt1).$$

We write $Z = \mathrm{rn}(\mathfrak{A}, X)$. $\mathfrak{A}$ *accepts* X iff $Ze \in K$.

3) Similarly, a *nondeterministic tree automaton over* Σ_n is $\mathfrak{A} = \langle \Sigma_k, I, L, K \rangle$, where $I \subseteq \Sigma_k$ is the set of initial states, and $L \subseteq \Sigma_n \times \Sigma_k \times \Sigma_k \times \Sigma_k$ is the

transition relation. A *run* of $\mathfrak{A}$ over the Σ_n-tree $X: U \to \Sigma_n$ is any Σ_k-tree $Z: U^+ \to \Sigma_k$ satisfying

$$(\forall t \in \mathrm{Br}^+(U))Zt \in I \wedge (\forall t \in U)(Xt, Zt, Zt0, Zt1) \in L.$$

We write $Z \in \mathrm{Rn}(\mathfrak{A}, X)$. $\mathfrak{A}$ *accepts* X iff there exists a run Z of $\mathfrak{A}$ over X such that $Ze \in K$.

4) A set of Σ_n-trees is *automaton definable* iff there is a tree automaton over Σ_n which accepts exactly the trees of the set.

Thus, tree automata are generalized in the natural way from the case of one successor. There is, however, one striking difference: Tree automata run down the tree, i.e. they start reading the input tree at its border and end up at the root. (For this reason they have to be 0-shift automata, i.e. the state at "time" t depends on the input at the same "time", whereas in the 1-shift automata of the linear case the state at time t depends on the input at the previous time.) The reason is that upward deterministic tree automata are rather weak, since at any point they carry the same information to both successors. (For an example, see Magidor–Moran [5], end of section 1.) It was for this reason that Doner [4], Magidor–Moran [5], and Thatcher–Wright [9] invented downward automata. (As a matter of fact, nondeterministic tree automata do not prefer a direction; we think of them as running downwards just for analogy.)

Tree automata share with ordinary automata the following facts, which we shall use: to any nondeterministic automaton there is an equivalent deterministic one; the automaton definable sets form a Boolean algebra; the emptiness problem is solvable. For more information about tree automata see Doner [4], Magidor–Moran [5], Thatcher–Wright [9], and Rabin [6].

As remarked in the introduction, the decison procedure for W2S presented here will follow closely the decision procedure for the Sequential Calculus SC of Buchi [2], as discussed by the author in [8]. The presentation here will be self-contained, but we will refer to [8] for proofs and for explanation of the methods used.

We identify Σ_n-trees with n-tuples of finite subsets of T_2 (monadic predicates restricted to a common finite transitive set), in a manner analogous to Buchi [1]. Thus, we can represent in the language of W2S the conditions specifying a tree automaton by propositional formulae involving set variables. We use X, Y, Z, sometimes with the upper index n, for n-tuples of set variables, i.e. for Σ_n-trees. The letters U, V, W will be used as before for set variables. s, s_i will denote tuples of truth values, $\bar{F}$ or $\bar{F}^n$ the tuple consisting of F's only. In this way we use formulae of the following three normal forms as tree automata in W2S (for details see [8], pp. 25 ff. and 87 ff.):

DEFINITION. *Automata normal forms* are the following:

Σ_R^0: $(\exists Z).(\forall t \in \mathrm{Br}^+(U))Zt = s_0 \wedge (\forall t \in U)Zt = \bar{J}[Xt, Zt0, Zt1] \wedge K[Ze]$

Σ^0: $(\exists Z).(\forall t \in \mathrm{Br}^+(U))I[Zt] \wedge (\forall t \in U)L[Xt, Zt, Zt0, Zt1] \wedge K[Ze]$

Σ_n^ω: $(\Delta Y)(\exists Z).K[Ze] \wedge (\forall t)L[Xt, Yt, Zt, Zt0, Zt1]$.

Here, I, K, L are propositional formulae involving at most the indicated prime formulae; $\bar{J}$ is a tuple of propositional formulae. (ΔY) is a string of $n-1$ alternating blocks of set quantifiers where the last one is universal.

Obviously, a Σ_R^0-formula or a Σ^0-formula is true for some X and some transitive U if and only if the corresponding deterministic, respectively non-deterministic, tree automaton accepts the tree $X \upharpoonright U$ (the function X restricted to U). A Σ_n^ω-formula corresponds to a nondeterministic automaton working (upwards!) on the infinite input X; X of course is constant outside a finite set. In view of this strong connection we will use automata terminology for Σ_R^0- and Σ^0-formulae, as e.g. $Z = \mathrm{rn}(\Phi, X \upharpoonright U)$ or $Z \in \mathrm{Rn}(\Phi, X \upharpoonright U)$ where Φ is an Σ_R^0-formula or a Σ^0-formula respectively.

The decidability proof will consist of three parts, which are the same as in the linear case, see e.g. [8], p. 23/24:

(i) Show that any formula of W2S not containing free individual variables can be transformed into the normal form Σ_n^ω for some n.

(ii) Show that Σ_1^ω-sentences are decidable.

(iii) Show that the negation of a Σ_1^ω-formula can be transformed into Σ_1^ω-form.

To prove that Σ_n^ω is a normal form for W2S, we need the following lemma. The idea of the proof is to restrict the consideration to a (finite) transitive set U which contains all the sets involved, and to use a set which is "stable" within U (see [8], p. 8). It is typical for proofs on Σ_n^ω to proceed this way.

LEMMA 4.1. *Σ_1^ω is closed under disjunction.*

PROOF. By Proposition 3.3(e) and (f),

$$\bigvee_{i=1}^{2} \{K_i[Z_ie] \wedge (\forall t)L_i[X_it, Z_it, Z_it0, Z_it1]\}$$

is equivalent to

$$(\exists U \in \text{Trans})\left\{(\forall t \notin U) \bigwedge_{i=1}^{2} [\neg Z_i t \wedge \neg X_i t]\right.$$

$$\left.\wedge \bigvee_{i=1}^{2} [K_i(e) \wedge (\forall t \in U) L_i(t) \wedge L_i[\bar{F}, \bar{F}, \bar{F}, \bar{F}]]\right\}.$$

Using the idea of lemma 7 of Buchi [2], the second half of the matrix of this formula is equivalent to

$$(\exists W)\{[We \wedge K_1(e) \wedge L_1[\bar{F}, \bar{F}, \bar{F}, \bar{F}]] \vee [\neg We \wedge K_2(e)$$

$$\wedge L_2[\bar{F}, \bar{F}, \bar{F}, \bar{F}]\} \wedge (\forall t \in U)\{[Wt \leftrightarrow Wt0] \wedge [Wt \leftrightarrow Wt1]$$

$$\wedge \{[Wt \wedge L_1(t)] \vee [\neg Wt \wedge L_2(t)]\}\}.$$

Putting these two equivalences together and using the definition of Trans, one sees that the disjunction of two Σ_1^ω-formulae can be written in Σ_1^ω-form.

Q.E.D.

THEOREM 4.2. *Any formula not containing free individual variables is equivalent to a Σ_n^ω-formula for a suitable n.*

The proof is the same as for the corresponding theorem I.1.d.1 of [8], pp. 20–23. We have to use Lemma 3.7 to eliminate conjunctions of existential individual quantifications, and the above lemma to reduce the number of disjunctions in the disjunctive normal form.

To be able to switch back and forth between deterministic and nondeterministic automata, we have to prove that for a Σ_R^0-formula to any input there exists a unique run. This is implicit in the notation, so in reading the next proposition the reader should recall that we prove derivability, not truth. Automaton recursion downward the tree bases on bar induction (Lemma 3.8), and thus is a trivial case of bar recursion.

PROPOSITION 4.3. *Bar recursion: For any Σ_R^0-formula Φ*

(a) $\text{Trans}(U) \wedge Z_1 = \text{rn}(\Phi, X \upharpoonright U) \wedge Z_2 = \text{rn}(\Phi, X \upharpoonright U) \rightarrow (\forall x \in U)[Z_1 x \leftrightarrow Z_2 x]$,

(b) $\text{Trans}(U) \rightarrow (\exists Z) Z = \text{rn}(\Phi, X \upharpoonright U)$.

The proof is analogous to the proof of Lemmata I.1.b.1 + 2, pp. 10–11 of [8]; it uses bar induction, Lemma 3.8, and set induction. Proposition 4.3 can be easily generalized to more general forms of recursions; we will, however, need only this form. Also we do not state the corresponding proposition on upward recursion.

It is by bar recursion, together with bar induction and induction on frontiered trees, Proposition 3.4, that we avoid Doner's tree induction and tree recursion ([4], p. 409). Doner's principles are not expressible in the language of W2S. Indeed, even the notion $\tau \upharpoonright w$, "the subtree of τ beginning at w" (Doner, l.c.), would make W2S undecidable, since it allows one to define concatenation.

THEOREM 4.4. *To any Σ^0-formula there is an equivalent Σ^0_R-formula.*

The proof is essentially the same as in the linear case, cf. theorem I.2.c.2 on p. 38 of [8]. Since the run of the Σ^0_R-formula is constructed by bar recursion, Proposition 4.3, the equivalence has to be proved by bar induction, Lemma 3.8.

COROLLARY 4.5. *Σ^0 is closed under Boolean operations.*

PROOF. As in the linear case, using Theorem 4.4 for negation. See e.g. [8], p. 34. Q.E.D.

To derive from Corollary 4.5 our main theorem, that Σ^ω_1 is closed under negation, we have to use the fact that Σ^ω_1 is decidable. This follows directly from the following construction, which is due to Rabin [7], proof of theor. 23:

Let Φ be a sentence in Σ^ω_1 containing k set quantifiers,

$$\Phi \underset{\mathrm{df}}{\equiv} (\exists Z^k)K[Ze] \wedge (\forall t)L[Zt, Zt0, Zt1].$$

Define sets $R_i \subseteq \Sigma_k$ as follows:

$R_0 =_{\mathrm{df}} \{\bar{F}^k\}$ iff $L[\bar{F}, \bar{F}, \bar{F}]$ holds; otherwise $R_0 =_{\mathrm{df}} \varnothing$. $R_{i+1} =_{\mathrm{df}} R_i \cup \{s \in \Sigma_k;$ ex. $s_0, s_1 \in R_i$ s.t. $L[s, s_0, s_1]$ holds$\}$. Since $R_i \subseteq R_{i+1}$ for all i, there is an $m \leqq 2^k$ such that $R_i = R_m$ for all $i \geqq m$. Let $R =_{\mathrm{df}} \bigcup_{i=0}^m R_i$.

LEMMA 4.6. $s \in R \leftrightarrow (\exists Z)\{Zx = s \wedge (\forall t \in T_x)L[Zt, Zt0, Zt1]\}$.

PROOF. Write $\psi_s(x)$ short for the right side of the lemma.

$\rightarrow$: We will show by (metamathematical) induction on i:

$$s \in R_i \rightarrow (\forall x)\psi_s(x).$$

This is trivial for $i = 0$ by axiom (OS). So let it be proven for i, let $s \in R_{i+1}$. Either $s \in R_i$, then we can use the induction hypothesis. Or else there are $s_0, s_1 \in R_i$ such that $L[s, s_0, s_1]$ holds. Let x be given. By the induction hypothesis $\psi_{s_0}(x0)$ and $\psi_{s_1}(x1)$ are true. Let Z_0 and Z_1 be the respective runs. Using Proposition 3.3(e) and (d), we define a run Z for $\psi_s(x)$ by ($\mathrm{COMP_{fin}}$). For $k = 1$ and $s = T$(true), the formula defining Z would be

$$t \in \mathrm{Cl}(Z_0) \cup \mathrm{Cl}(Z_1) \wedge \{[x0 \leqq t \wedge Z_0 t] \vee [x1 \leqq t \wedge Z_1 t] \vee t = x\}.$$

For $s = F$(false), the clause $t = x$ would be dropped. For arbitrary k one has to use k formulae to define the k components of Z.

$\leftarrow$: By Proposition 3.3(e), $\psi_s(x)$ implies that there is a transitive set U such that $x \in U$ and

$$(\exists Z \subseteq U) \cdot Zx = s \wedge (\forall t \in T_x) L[Zt, Zt0, Zt1].$$

So let U be transitve. We will prove by bar induction on

$$\psi(x) \underset{\mathrm{df}}{\equiv} \bigwedge_{s \in \Sigma_k} [(\exists Z \subseteq U)\{Zx = s \wedge (\forall t \in T_x) L[Zt, Zt0, Zt1]\} \rightarrow s \in R]$$

that $(\forall x \in U^+)\psi(x)$. The lemma will follow.

For $x \in \mathrm{Br}^+(U)$, $\psi(x)$ is easily seen to be true. Let $x \in U$, $s \in \Sigma_k$, let Z be such that

$$Z \subseteq U \wedge Zx = s \wedge (\forall t \in T_x) L[Zt, Zt0, Zt1].$$

Let $s_0 = Zx0$, $s_1 = Zx1$. Then for $j = 0, 1$,

$$Z \subseteq U \wedge Zxj = s_j \wedge (\forall t \in T_{xj}) L[Zt, Zt0, Zt1].$$

Thus, $s_0, s_1 \in R$ by the induction hypothesis. Since $L[s, s_0, s_1]$, we have $s \in R$.

Q.E.D.

Since the set R is computable, we get:

THEOREM 4.7. *Σ_1^ω is decidable. In fact, for any sentence Φ in Σ_1^ω we can effectively construct a derivation of either Φ or $\neg\Phi$.*

Note that Lemma 4.6 implies that all subtrees T_x are "isomorphic relative to input free automata", i.e.

$$(1) \qquad (\exists Z)\{Zx = s \wedge (\forall t \in T_x) L[Zt, Zt0, Zt1]\}$$

$$\leftrightarrow (\exists Z)\{Zy = s \wedge (\forall t \in T_y) L[Zt, Zt0, Zt1]\}.$$

By relativizing the completeness proof for W2S, the "isomorphism relative to W2S-sentences" is also derivable, i.e.

$$(2) \qquad \Phi(x)^{(T_x)} \leftrightarrow \Phi(y)^{(T_y)}$$

for any formula $\Phi(x)$ containing x as the only free variable and not containing the constant e. (Here $\Phi^{(T_x)}$ is the relativization of all quantifiers in Φ to T_x.) Note that (2) cannot be extended to formulae containing other free variables, since within W2S we cannot map elements or subsets of T_x into the corresponding elements or subsets of T_y.

The known proofs for the decidability of Σ_1^ω used the fact that, if an automaton admits a run at all then it admits a "short" one (see e.g. Doner [4], p. 413). To formalize this proof one needs (1) to cut down a given run which is too long. Also one needs a stronger version of Lemma 4.6, which is more cumbersome to derive. The recursive character of the construction of Rabin is better suited for our inductive proofs.

We need a further lemma:

LEMMA 4.8. $\mathrm{Fr}(U) \wedge (\forall x \in U)(\exists Z)\{K[Zx] \wedge (\forall t \in T_x)L[Zt, Zt0, Zt1]\} \rightarrow (\exists Z)\{(\forall x \in U)K[Zx] \wedge (\forall t \notin U^-)L[Zt, Zt0, Zt1]\}$:
If one can start a given automaton on every point of a frontier, then there is a single run from which one can get all the separate runs by restrictions.

PROOF. It is easy to prove by set induction and Proposition 3.3(e)

$$(\forall x, y \in U)[x \sim y \rightarrow x = y] \wedge (\forall x \in U)(\exists Z)\{K[Zx] \wedge$$
$$\wedge (\forall t \in T_x)L[Zt, Zt0, Zt1]\} \rightarrow (\exists Z)(\forall x \in U)\{K[Zx] \wedge$$
$$\wedge (\forall t \in T_x)L[Zt, Zt0, Zt1]\}.$$

This directly implies the lemma. Q.E.D.

THEOREM 4.9. *Σ_1^ω is closed under Boolean operations.*

PROOF. Conjunction is easy. So let $\Phi(X) \in \Sigma_1^\omega$ be the formula

$$(\exists Z) \cdot K[Ze] \wedge (\forall t)L[Xt, Zt, Zt0, Zt1].$$

By restricting the consideration to $\mathrm{Cl}(X)$ as in the proof of Lemma 4.1, we see that $\Phi(X)$ is equivalent to

$$\begin{aligned}(\forall U \in \mathrm{Trans})\{(\forall t \notin U) \neg Xt \rightarrow (\exists Z)\{K[Ze] \\ \wedge (\forall t \in U)L[Xt, Zt, Zt0, Zt1] \wedge (\forall t \notin U)L[\bar{F}, Zt, Zt0, Zt1]\}\}.\end{aligned} \tag{1}$$

Using the formula $L[\bar{F}, Zt, Zt0, Zt1]$ we define the set R as in Lemma 4.6, and construct a propositional formula I s.t. $I[s] \leftrightarrow s \in R$. Then (1) is equivalent to

$$\begin{aligned}(\forall U \in \mathrm{Trans})\{(\forall t \notin U) \neg Xt \rightarrow (\exists Z)\{K[Ze] \wedge \\ \wedge (\forall t \in U)L[Xt, Zt, Zt0, Zt1] \wedge (\forall t \in \mathrm{Br}^+(U))I[Zt]\}\}.\end{aligned} \tag{2}$$

Indeed, (1)→(2) is immediate from Lemma 4.6.

For (2)→(1) use Lemmata 4.6 and 4.8 together with Proposition 3.3(e). The second half of (2) is a Σ^0-formula $\psi_1(X, U)$. Thus by Corollary 4.5 there is a Σ^0-formula $\psi_2(X, U)$ equivalent to $\neg\psi_1(X, U)$. Therefore $\neg\Phi(X)$ is equivalent to

$$(\exists U \in \mathrm{Trans})\{(\forall t \notin U) \neg Xt \wedge \psi_2(X, U)\},$$

which is easily transformed into Σ_1^ω. Q.E.D.

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